SEPARATION THEOREMS FOR DIFFERENTIAL EQUATIONS OF THE THIRD AND FOURTH ORDER

BY

JAMES RONALD FRASER KENT

A.B., Queen's University, 1933

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AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1947

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I. INTRODUCTION

The present paper establishes certain results concerning the separation of the zeros of real-valued solutions of ordinary linear homogeneous differential equations of the third and fourth order.* The methods are general and will be demonstrated in detail in the case of the third order equation. For equations of order higher than the fourth they become somewhat cumbersome and sufficiently restrictive that the applicability of the results would appear to be rather limited.

II. EQUATIONS OF THE THIRD ORDER

1. Consider the linear homogeneous equation of the third order:

$$y''' + p y'' + q y' + r y = 0, \tag{1}$$

in which the derivatives are assumed to be with respect to a single real variable x, and where the coefficients p, q, r are real-valued, single-valued, continuous and differentiable functions of x only. Let y_1, y_2 be any two solutions of (1), and let their Wronskian $y_1 y_2' - y_1' y_2$ be denoted by u. If y_1 and y_2 are linearly independent solutions,

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*The second order equation has been examined in classical theory by Sturm and others; see Bocher, Lecons sur les Methodes de Sturm, Paris, 1917, p.45. For a "general separation theorem" involving two linearly independent solutions of an equation of higher order and their Wronskian, see Birkhoff, Annals of Math., ser.2, vol.12 (1911), p.109, and C. N. Reynolds, Trans. Amer. Math. Soc., vol.22 (1921), p.221.

u is not identically zero.

By successive differentiation it is easily shown that u satisfies the "associate equation:"

$$(2) \qquad u''' + 2pu'' + (p^2 + p' + q)u' + (pq + q' - r)u = 0 .$$

Suppose now that there exists a value a of x for which

$$(3) \qquad p \leqq 0, \quad q \leqq -p^2 - p', \quad r \geqq pq + q' ;$$

then at a each of the coefficients in (2) after the first is $\leqq 0$; and if u be a real-valued solution of (2) defined by the boundary values:

$$(4) \qquad u^{(i)}(a) = \gamma_i > 0 , \qquad i = 0, 1, 2 ,$$

it is evident from (2) that $u'''(a) \geqq 0$.

Now, by the law of the mean for continuous functions, for any $u^{(i)}$ $(i=0,1,2)$ which is continuous in a closed interval $a \leqq x \leqq b$ we have

$$u^{(i)}(b) - u^{(i)}(a) = (b-a)\, u^{(i+1)}(\xi) , \qquad a < \xi < b .$$

Hence, by (4), if any $u^{(i)}$ vanishes at $x = b$, $u^{(i+1)}(\xi) < 0$; that is, between a and any zero of $u^{(i)}$ which is greater than a , $u^{(i+1)}$ becomes negative. But, by equation (2), if there exists any interval $a \leqq x \leqq b$ throughout which the inequalities (3) remain valid, then in this interval u''' is never negative. Consequently each $u^{(i)}$ $(i=0,1,2)$, and in particular u itself, does not vanish in this interval.

If, therefore, the existence of two <u>real-valued</u>

solutions of equation (1) which have for Wronskian the above function u can be demonstrated, the zeros of these two solutions must separate each other throughout the interval $a \leqq x \leqq b$. Letting y_λ, y_μ be two continuous solutions of (1), having respectively the boundary values

$$y_\lambda^{(j)}(a) = \lambda_j \,, \quad y_\mu^{(j)}(a) = \mu_j \,, \qquad j = 0, 1, 2, \tag{5}$$

and having for Wronskian the function u defined by (2) and (4), it may be shown that, by a suitable selection of positive real values for the r_i, corresponding real values of the λ_j, μ_j exist. (The λ_j, and likewise the μ_j, are then, however, linearly dependent - see §2, below.) Hence we have:

Theorem 1. If there exists any closed interval $a \leqq x \leqq b$ throughout which

$$p \leqq 0 \,, \quad q \leqq -p^2 - p' \,, \quad r \geqq pq + q' \,, \tag{3}$$

then there exist pairs of real-valued, continuous solutions of the differential equation of the third order:

$$y''' + py'' + qy' + ry = 0 \tag{1}$$

which are such that the zeros of the solutions of each pair separate each other throughout the interval.

Corollary 1. If throughout any closed interval $a \leqq x \leqq b$,

$$q \leqq 0 \,, \quad r \geqq q' \,,$$

then the reduced third order equation $y''' + qy' + ry = 0$

possesses pairs of real-valued solutions whose zeros separate each other throughout this interval.

Corollary 2. If there exists any closed interval throughout which the coefficients of y and y' are non-positive, the self-adjoint third order equation, viz.:

$$y''' + q y' + \tfrac{1}{2} q' y = 0$$

possesses pairs of real-valued solutions whose zeros separate each other throughout this interval.

2. We now examine the conditions under which, given a particular real-valued solution of equation (1), there will exist another real-valued solution whose zeros alternate with those of the given one. This may be done most conveniently by an analysis of the types of solutions represented by the functions y_λ, y_μ defined by (1) and (5), the λ_j, μ_j being restricted by the relations

$$(6) \qquad \begin{cases} \lambda_0\mu_1 - \lambda_1\mu_0 = \gamma_0 \ , \quad \lambda_0\mu_2 - \lambda_2\mu_0 = \gamma_1 \ , \\ \lambda_1\mu_2 - \lambda_2\mu_1 = \gamma_2 + \gamma_1 p(a) + \gamma_0 q(a). \end{cases}$$

It is found that, of those real-valued solutions which vanish at $x=a$, y_λ (or y_μ) represents those, and only those, whose first and second derivatives are of like sign ($\neq 0$) at a, and that of those which do not vanish at a, it represents those, and only those, for which

either $y'(a)/y(a) > p(a)$ or $y''(a)/y(a) < -q(a)$

(or both).

Hence, in view of Theorem 1, we have further:

Theorem 2. Given the third order differential equation

(1) $$y''' + py'' + qy' + ry = 0$$

and any closed interval $a \leqq x \leqq b$ throughout which

(3) $$p \leq 0, \quad q \leqq -p^2 - p', \quad r \geqq pq + q';$$

then if y_λ be any real-valued solution of (1) such that either

(i) $y_\lambda(a) = 0$ and $y_\lambda'(a) \cdot y_\lambda''(a) > 0$, or

(ii) $y_\lambda(a) \neq 0$ and $y_\lambda'(a)/y_\lambda(a) > p(a)$, or

(iii) $y_\lambda(a) \neq 0$ and $y_\lambda''(a)/y_\lambda(a) < -q(a)$,

there exists another real-valued solution y_μ of (1) which vanishes alternately with y_λ throughout this interval.

Similar statements might be made in respect of Corollaries 1 and 2 concerning the reduced and the self-adjoint equations of the third order.

It is pointed out that the "alternating solution" is by no means unique. For, in addition to the trivial fact that obviously any solution linearly dependent upon an alternating solution will be an alternating solution, it will also be observed, upon solving equations (6) for the μ_j in terms of the λ_j, that since some $\lambda_\kappa \neq 0$ $(\kappa = 0, 1, 2)$, μ_κ is an arbitrary element.

III. THE SELF-ADJOINT EQUATION OF THE FOURTH ORDER

1. Examining the self-adjoint fourth order equation,

$$y'''' + qy'' + q'y' + sy = 0, \tag{7}$$

by the above methods, we first find that the "associate equation" is of the fifth order, viz :

$$u^{v} + 2qu''' + 3q'u'' + (q^2 + 3q'' - 4s)u' + (q''' + qq' - 2s')u = 0. \tag{8}$$

Corresponding to a solution u of this equation defined by the boundary values $u^{(i)}(a) = \gamma_i > 0 \quad (i = 0, 1, 2, 3, 4)$, it can readily be shown that there exist pairs of real-valued solutions of (7) having u for Wronskian, provided the γ_i are so chosen that

$$\begin{aligned} &\gamma_2^2 - 2\gamma_1\gamma_3 + 2\gamma_0\gamma_4 + \gamma_0^2\left[q^2(a) + 2q''(a) - 4s(a)\right] \\ &\quad - 2\gamma_1^2 q(a) + 4\gamma_0\gamma_2 q(a) + 2\gamma_0\gamma_1 q'(a) \geqq 0 . \end{aligned} \tag{9}$$

This latter condition is, however, easily satisfied - e.g., by a suitably large choice of γ_4 subsequent to the choice of the other γ's. Hence we have:

Theorem 3. If throughout any closed interval $a \leqq x \leqq b$:

$$q, q' \leqq 0, \quad s \geqq \tfrac{1}{4}(q^2 + 3q''), \quad s' \geqq \tfrac{1}{2}(qq' + q'''), \tag{10}$$

then there exist pairs of real-valued, continuous solutions of the self-adjoint fourth order equation

$$y'''' + qy'' + q'y' + sy = 0 \tag{7}$$

which are such that the solutions of each pair vanish alternately

throughout the interval.

2. A slight extension of this theorem may be obtained as follows:

The value of the left-hand member of inequality (9) may be shown to be, subsequent to the choice of the γ_i, independent of a. For if the associate equation (8) be multiplied by u and integrated, there results:

$$u''^2 - 2u'u''' + 2uu'^{iv} + 4q\,uu'' - 2q\,u'^2 + 2q'uu' + 2q''u^2 + q^2u^2 - 4su^2 = C, \text{ a constant};$$

and if one compares the left side of this relation with that of inequality (9), recalling that each $\gamma_i = u^{(i)}(a)$, it is seen that they are identical at a. Employing this fact we may now, by solving relations similar to those of (6) for the third order equation, set up pairs of solutions y_λ, y_μ of (7) having the boundary values:

$$(11)\quad \begin{cases} j: & 0, & 1, & 2, & 3 \\ y_\lambda^{(j)}(a): & 1, & 0, & \frac{1}{2\gamma_0}\left[-\gamma_2 - \gamma_0 q(a) \pm \sqrt{C}\right], & -\frac{1}{2\gamma_0}\left[\gamma_3 + \gamma_1 q(a) + \gamma_0 q'(a)\right] \\ y_\mu^{(j)}(a): & 0, & \gamma_0, & \gamma_1, & \frac{1}{2}\left[\gamma_2 - \gamma_0 q(a) \pm \sqrt{C}\right]. \end{cases}$$

These solutions will be real-valued if $C \geq 0$ (cf. inequality (9)); and if C is actually > 0 — a condition which is easily met by a suitable choice of the γ_i — we have defined by (7) and (11) four distinct real-valued solutions. The Wronskian of these four solutions can be shown to have at $x = a$ the value $C\,(\neq 0)$; and since the Wronskian of any n solutions of the nth order differential equation

$$y^{(n)} + \sum_{\nu=2}^{n} p_\nu y^{(n-\nu)} = 0$$

is a constant, these four solutions are consequently linearly

independent. Theorem 3 may therefore be amplified to read as follows:

Theorem 4. If throughout any closed interval $a \leqq x \leqq b$:

(10) $q, q' \leqq 0, \quad s \geqq \frac{1}{4}(q^2 + 3q''), \quad s' \geqq \frac{1}{2}(qq' + q'''),$

then there exist four linearly independent, real-valued, continuous solutions, $y_{\lambda_1}, y_{\lambda_2}, y_{\mu_1}, y_{\mu_2}$, of the equation

$$y'''' + q y'' + q' y' + s y = 0 \tag{7}$$

which are such that throughout the interval the zeros of each y_{λ_κ} separate those of the corresponding y_{μ_κ} $(\kappa = 1, 2)$.

3. As in the case of the third order equation, we may also enquire as to the conditions under which, corresponding to a given real-valued solution of (7), there exists another real-valued solution whose zeros alternate with those of the given one. At least a partial answer to this question may be obtained (as before) by an analysis of the types of solutions which can be represented respectively by the functions y_λ, y_μ defined by (7) and (11). (Other interlacing solutions, with neither $y(a)$ nor $y'(a) = 0$, may of course also exist.) From such an analysis it is found that:

Theorem 5. If at $x = a$

(10) $q, q' \leqq 0, \quad s \geqq \frac{1}{4}(q^2 + 3q''), \quad s' \geqq \frac{1}{2}(qq' + q'''),$

then to any given real-valued solution $\bar{y}$ of the self-adjoint

fourth order equation

(7) $$y'''' + qy'' + q'y' + sy = 0$$

which is such that $\bar{y}(a) = 0$ and $\bar{y}'(a) \cdot \bar{y}''(a) > 0$ there corresponds another real-valued solution whose zeros alternate with those of the given one throughout any closed interval $a \leqq x \leqq b$ in which the inequalities (10) remain valid.

and

Theorem 6. If at $x = a$

(10) $$q, q' \leqq 0, \quad s \geqq \tfrac{1}{4}(q^2 + 3q''), \quad s' \geqq \tfrac{1}{2}(qq' + q'''),$$

then to any given real-valued solution $\bar{y}$ of the self-adjoint fourth order equation

(7) $$y'''' + qy'' + q'y' + sy = 0$$

which is such that

(12) $$\bar{y}'(a) = 0, \quad \bar{y}(a) \cdot \bar{y}''(a) > 0, \quad \bar{y}(a) \cdot \bar{y}'''(a) < 0,$$

there corresponds another real-valued solution, which vanishes at a and thereafter alternately with the given one throughout any closed interval $a \leqq x \leqq b$ in which the inequalities (10) remain valid.

In case $q(a) < 0$, conditions (12) may be relaxed to read:

$$\bar{y}(a) \neq 0, \quad \bar{y}'(a) = 0, \quad \bar{y}(a) \cdot \bar{y}''(a) \geqq 0.$$

IV. THE NON-SELF-ADJOINT FOURTH ORDER EQUATION

We assume this equation to be written in the form

$$y^{IV} + qy'' + \lambda y' + sy = 0 ;$$

or, since the case $\lambda \equiv q'$ yields the self-adjoint equation, already considered, we may by introducing the function $t \equiv q' - \lambda$ discuss the non-self-adjoint case more conveniently in the form:

$$y^{IV} + qy'' + (q' - t)y' + sy = 0, \qquad t \not\equiv 0. \tag{13}$$

The associate equation in this case is found to be of the sixth order, viz :

$$\begin{cases} tu^{VI} + t'u^{V} + 2qtu^{IV} + (5q't^2 + 2qt')u''' \\ \quad + (q^2t + 6q''t + 3q't' - 4tt' - 4ts)u'' \\ \quad\quad + (q^2t' + 3qq't + 3q''t' + 4q'''t - 3tt'' \\ \quad\quad\quad - 2t'^2 - 6ts' - 4t's)u' + (qq't' + qq''t \\ \quad\quad\quad\quad - 2qtt' + q'^2t - t^3 + q^{IV}t - tt''' \\ \quad\quad\quad\quad\quad - 2ts'' + q'''t' - t't'' - 2t's')u = 0 . \end{cases} \tag{14}$$

As before, selecting a solution u of this equation defined by boundary values $u^{(i)}(a) = \gamma_i > 0$, $i = 0, 1, 2, 3, 4, 5$, we may demonstrate the existence of pairs of real-valued solutions of (13) which have the function u for Wronskian, and which will therefore vanish alternately throughout any interval in which u does not vanish. Such an interval may be specified in terms of inequality conditions on the coefficients of equation (14). We have, therefore, finally:

Theorem 7. Given the non-self-adjoint differential equation of the fourth order:

(13) $$y'''' + qy'' + (q' - t)y' + sy = 0, \qquad t \neq 0,$$

and any closed interval $a \leqq x \leqq b$ in which $q < 0$, t does not vanish, and each of the functions

(15) $$\begin{cases} t', \ 5q't^2 + 2qt', \ q^2t + 6q''t + 3q't' - 4tt' - 4ts, \\ q^2t' + 3qq't + 3q''t' + 4q'''t - 3tt'' - 2t'^2 - 6ts' - 4t's, \\ qq't' + qq''t - 2qtt' + q'^2t - t^3 + q''''t - tt''' - 2ts'' \\ \quad + q'''t' - t't'' - 2t's' \end{cases}$$

either vanishes or is of sign opposite to that of t; then there exist pairs of real-valued, continuous solutions of (13) such that the solutions of each pair vanish alternately throughout the interval.

(As evidence of the fact that the conditions of this theorem are not unduly restrictive, one might cite the case of an equation with coefficients q and t both negative constants. The conditions of the theorem are then satisfied throughout any interval in which

$$s \geqq \tfrac{1}{4}q^2, \qquad s' \geqq 0, \qquad s'' \geqq -\tfrac{1}{2}t^2.$$

For example, if s is any positive constant $\geqq \frac{1}{4}q^2$ the theorem holds throughout any interval of the finite x-axis; or if s is any monotone increasing function of x bounded below by $\frac{1}{4}q^2$, it holds for all intervals throughout which the derived function s' is "almost" monotone increasing ($s'' \geqq -\frac{1}{2}t^2$) or actually so; etc.)

VITA

James Ronald Fraser Kent was born in Halifax, Nova Scotia, on February 29, 1912. He attended the public schools of Halifax and Kingston, Ontario, and entered Queen's University, Kingston, in 1929. He received from Queen's University the degrees of Bachelor of Arts in 1933 and Master of Arts in 1934. During the following academic year he attended Syracuse University under a graduate scholarship, and was elected an associate member of the Society of the Sigma Xi. In 1935-39 he was a graduate assistant at the University of Illinois, following which he held an instructorship at the University of Arkansas and also instructed in summer sessions at Syracuse University. In 1942 he entered the Royal Canadian Navy, serving for a time at sea, and later as Assistant Director of Naval Education with the rank of Lieutenant Commander. In 1946 he was appointed an assistant professor of mathematics at the University of British Columbia.